

# Some applications of modular units

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## Abstract

We show that a weakly holomorphic modular function can be written as a sum of modular units of higher level. We further find a necessary and sufficient condition for a Siegel modular function of degree  $g$  to have neither zero nor pole on the domain when restricted to certain subset of the Siegel upper half-space  $\mathbb{H}_g$ .

## 1 Introduction

Let  $g$  be a positive integer. We let

$$\mathbb{H}_g = \{Z \in \text{Mat}_g(\mathbb{C}) \mid {}^t Z = Z, \text{Im}(Z) \text{ is positive definite}\}$$

be the Siegel upper half-space of degree  $g$  on which the symplectic group

$$\text{Sp}_g(\mathbb{Z}) = \{\gamma \in \text{GL}_{2g}(\mathbb{Z}) \mid {}^t \gamma J \gamma = J\} \quad \text{with } J = \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix}$$

acts by the rule

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} (Z) = (AZ + B)(CZ + D)^{-1},$$

where  $A, B, C, D$  are  $g \times g$  block matrices. For a positive integer  $N$  we further let

$$\Gamma(N) = \{\gamma \in \text{Sp}_g(\mathbb{Z}) \mid \gamma \equiv I_{2g} \pmod{N}\}$$

be the congruence subgroup modulo  $N$  of the group  $\text{Sp}_g(\mathbb{Z})$ . In particular, when  $g = 1$ ,  $\mathbb{H}_g$  becomes the upper half-plane  $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$  and  $\text{Sp}_g(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})$  acts on it by fractional linear transformations.

Define a subset  $\mathbb{H}_g^{\text{diag}}$  of  $\mathbb{H}_g$  by

$$\mathbb{H}_g^{\text{diag}} = \{\text{diag}(\tau_1, \tau_2, \dots, \tau_g) \mid \tau_1, \tau_2, \dots, \tau_g \in \mathbb{H}\},$$

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where  $\text{diag}(\tau_1, \tau_2, \dots, \tau_g)$  stands for the  $g \times g$  diagonal matrix whose diagonal entries are  $\tau_1, \tau_2, \dots, \tau_g$ . If  $g = 1$ , then  $\mathbb{H}_g^{\text{diag}}$  is nothing but  $\mathbb{H}$ . Let  $f(Z)$  be a (meromorphic) Siegel modular function of degree  $g$  and level  $N$  (over  $\mathbb{C}$ ), namely  $f(Z)$  is a quotient of two Siegel modular forms of degree  $g$  and the same weight so that it is invariant under  $\Gamma(N)$ . When  $g = 1$ ,  $f$  becomes a usual meromorphic modular function of level  $N$ . We shall mainly consider the case where  $f$  has neither zero nor pole on  $\mathbb{H}_g^{\text{diag}}$ .

Let  $X(N) = \bar{\Gamma}(N) \backslash \mathbb{H}^*$  be the modular curve of level  $N$  that is a compact Riemann surface, where  $\bar{\Gamma}(N) = \Gamma(N) / \{\pm I_2\}$  and  $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ . We denote its function field by  $\mathbb{C}(X(N))$ . As is well-known,  $X(1)$  is of genus zero and  $\mathbb{C}(X(1)) = \mathbb{C}(j)$ , where

$$j = j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots \quad (q = e^{2\pi i\tau}, i = \sqrt{-1})$$

is the elliptic modular function [8, Theorem 2.9]. Furthermore,  $\mathbb{C}(X(N))$  is a Galois extension of  $\mathbb{C}(X(1))$  whose Galois group is naturally isomorphic to  $\bar{\Gamma}(1)/\bar{\Gamma}(N)$ . Let  $\mathcal{O}_N$  be the integral closure of  $\mathbb{C}[j]$  in  $\mathbb{C}(X(N))$ . We call the invertible elements in  $\mathcal{O}_N$  *modular units* of level  $N$  (over  $\mathbb{C}$ ), which are precisely those functions in  $\mathbb{C}(X(N))$  having no zeros and poles on  $\mathbb{H}$  [6, p.36]. Kubert and Lang developed in [6] the theory of modular units in terms of Siegel functions which will be defined in §2. (In addition, they require that the Fourier coefficients of a modular unit of level  $N$  lie in the  $N$ th cyclotomic field.) In this paper we shall first describe  $\mathcal{O}_N$  in view of modular units when  $N \equiv 0 \pmod{4}$  (Theorem 3.3), and then conclude that any weakly holomorphic modular function can be expressed as a sum of modular units of higher level (Corollary 3.5). Here, a function is said to be weakly holomorphic if it is holomorphic on  $\mathbb{H}$ .

On the other hand, suppose that  $g$  and  $N$  are two positive integers  $\geq 2$ , and let  $f(Z)$  be a Siegel modular function of degree  $g$  and level  $N$ . We shall further prove that  $f(Z)$  has neither zero nor pole on  $\mathbb{H}_g^{\text{diag}}$  if and only if  $f(\text{diag}(\tau_1, \tau_2, \dots, \tau_g))$  is a product of  $g$  modular units of variables  $\tau_1, \tau_2, \dots, \tau_g \in \mathbb{H}$ , respectively (Theorem 4.2). To this end we shall examine some necessary basic properties of modular units in §2. And, we shall show that certain quotient of theta constants of degree  $g$  on  $\mathbb{H}_g^{\text{diag}}$  is a product of modular units (Example 4.3).

## 2 Properties of modular units

For a positive integer  $N$  we denote the group of all modular units of level  $N$  by  $V_N$  (that is,  $V_N = \mathcal{O}_N^\times$ ), which contains  $\mathbb{C}^\times$  as a subgroup. In this section we shall develop some necessary properties about modular units which will be used in later sections.

**LEMMA 2.1.** *If  $f$  is a weakly holomorphic modular function of level 1, then it is a polynomial of  $j$  over  $\mathbb{C}$ , that is  $f \in \mathbb{C}[j]$ .*

**PROOF.** [7, Theorem 2]. □

**REMARK 2.2.** Note that  $j$  gives rise to a bijection  $j : \bar{\Gamma}(1) \backslash \mathbb{H} \rightarrow \mathbb{C}$  [7, Chapter 3 Theorem 4].

PROPOSITION 2.3. *Let  $h \in \mathbb{C}(X(N))$ . Then,  $h$  is weakly holomorphic if and only if  $h$  is integral over  $\mathbb{C}[j]$ .*

PROOF. Assume that  $h = h(\tau)$  is weakly holomorphic. We consider the following monic polynomial of  $X$

$$P(X) = \prod_{\gamma \in \overline{\Gamma}(1)/\overline{\Gamma}(N)} (X - h \circ \gamma).$$

Since  $\text{Gal}(\mathbb{C}(X(N))/\mathbb{C}(X(1))) \simeq \overline{\Gamma}(1)/\overline{\Gamma}(N)$ , every coefficient of  $P(X)$  belongs to  $\mathbb{C}(X(1))$  and is holomorphic on  $\mathbb{H}$ . So, it is a polynomial of  $j$  over  $\mathbb{C}$  by Lemma 2.1. This shows that  $h$  is integral over  $\mathbb{C}[j]$ .

Conversely, assume that  $h$  is integral over  $\mathbb{C}[j]$ . Then  $h$  is a zero of a monic polynomial

$$X^n + P_{n-1}(j)X^{n-1} + \cdots + P_1(j)X + P_0(j),$$

where  $n \geq 1$  and  $P_{n-1}(X), \dots, P_1(X), P_0(X) \in \mathbb{C}[j][X]$ . Suppose on the contrary that  $h$  has a pole at  $\tau_0 \in \mathbb{H}$  (so,  $h \neq 0$ ). Since  $h$  satisfies

$$h^n + P_{n-1}(j)h^{n-1} + \cdots + P_1(j)h + P_0(j) = 0,$$

we get by dividing both sides by  $h^n$  and substituting  $\tau = \tau_0$

$$1 + P_{n-1}(j(\tau_0))(1/h(\tau_0)) + \cdots + P_1(j(\tau_0))(1/h(\tau_0))^{n-1} + P_0(j(\tau_0))(1/h(\tau_0))^n = 0.$$

This yields a contradiction  $1 = 0$  because  $j(\tau_0) \in \mathbb{C}$  and  $1/h(\tau_0) = 0$ . Therefore  $h$  must be weakly holomorphic.  $\square$

REMARK 2.4. By definition,  $h \in \mathbb{C}(X(N))$  is a modular unit if and only if both  $h$  and  $h^{-1}$  are integral over  $\mathbb{C}[j]$ . Hence, Proposition 2.3 gives an elementary proof of the well-known fact that  $h$  is a modular unit if and only if it has no zeros and poles on  $\mathbb{H}$  [6, p.36].

Given a vector  $\begin{bmatrix} r \\ s \end{bmatrix} \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$  for  $N \geq 2$ , the Siegel function  $g_{\begin{bmatrix} r \\ s \end{bmatrix}}(\tau)$  is defined on  $\mathbb{H}$  by the following infinite product

$$g_{\begin{bmatrix} r \\ s \end{bmatrix}}(\tau) = -q^{(1/2)(r^2-r+1/6)} e^{\pi i s(r-1)} (1 - q^r e^{2\pi i s}) \prod_{n=1}^{\infty} (1 - q^{n+r} e^{2\pi i s})(1 - q^{n-r} e^{-2\pi i s}), \quad (1)$$

which is a weakly holomorphic modular function of level  $12N^2$  [6, Chapter 3 Theorem 5.2].

LEMMA 2.5. *Suppose  $N \geq 2$  and let  $n$  be the number of inequivalent cusps of  $X(N)$ . Then, the rank of the subgroup of  $V_N/\mathbb{C}^\times$  generated by  $g_{\begin{bmatrix} r \\ s \end{bmatrix}}(\tau)^{12N}$  for  $\begin{bmatrix} r \\ s \end{bmatrix} \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$  is  $n - 1$ .*

PROOF. [6, Chapter 2 Theorem 3.1].  $\square$

REMARK 2.6. We have the formula

$$n = |\bar{\Gamma}(1)/\bar{\Gamma}(N)|/N = \begin{cases} 3 & \text{if } N = 2, \\ (N^2/2) \prod_{p|N} (1 - p^{-2}) & \text{if } N > 2 \end{cases}$$

[8, pp.22–23].

PROPOSITION 2.7. *With the same assumption and notation as in Lemma 2.5,  $V_N/\mathbb{C}^\times$  is a free abelian group of rank  $n - 1$ .*

PROOF. Let  $\infty_1, \infty_2, \dots, \infty_n$  be the inequivalent cusps of  $X(N)$ , and let  $\mathcal{D}_N$  be the free abelian group of rank  $n$  generated by these cusps. Then, an element of  $\mathcal{D}_N$  is uniquely written as

$$m_1(\infty_1) + m_2(\infty_2) + \dots + m_n(\infty_n) \quad \text{for some integers } m_1, m_2, \dots, m_n.$$

Now, we consider a (well-defined) injective homomorphism

$$\begin{aligned} V_N/\mathbb{C}^\times &\rightarrow \mathcal{D}_N \\ h &\mapsto \text{div}(h). \end{aligned}$$

If  $h \in V_N/\mathbb{C}^\times$  has  $\text{div}(h) = \sum_{k=1}^n m_k(\infty_k)$ , then we get the relation  $\sum_{k=1}^n m_k = 0$ . Hence  $V_N/\mathbb{C}^\times$  is a free abelian group of rank  $\leq n - 1$ . Thus it follows from Lemma 2.5 that the rank of  $V_N/\mathbb{C}^\times$  is exactly  $n - 1$ .  $\square$

REMARK 2.8. Since every cusp of  $X(1)$  is equivalent to  $i\infty$  [8, p.14], if  $h \in V_1$  then  $\text{div}(h) = m(i\infty)$  for some integer  $m$ . On the other hand, now that the sum of the orders of zeros and poles of  $h$  is zero, we get  $m = 0$ . This yields  $V_1 = \mathbb{C}^\times$ .

LEMMA 2.9. *Let  $N \geq 2$  and  $h \in V_N - \mathbb{C}^\times$ . There is a finite subset  $S$  of  $\mathbb{C}^\times$  so that the map*

$$\begin{aligned} \varphi : \mathbb{H} &\rightarrow \mathbb{C}^\times - S \\ \tau &\mapsto h(\tau) \end{aligned}$$

*is surjective.*

PROOF. Consider the following holomorphic map between compact Riemann surfaces

$$\begin{aligned} X(N) &\rightarrow \mathbb{P}^1(\mathbb{C}) \\ \tau &\mapsto [h(\tau) : 1]. \end{aligned}$$

Since  $h$  is not a constant, the above map is surjective. Take a subset  $S$  of  $\mathbb{C}^\times$  as

$$S = \{h(\tau) \mid \tau \text{ is a cusp of } X(N)\} - \{0, \infty, h(\tau) \mid \tau \in \mathbb{H}\}.$$

Since there are only finitely many inequivalent cusps of  $X(N)$ , it is a finite set. And, the map  $\varphi$  becomes surjective.  $\square$

PROPOSITION 2.10. *Let  $h$  be a modular unit. Suppose that*

$$\text{ord}_q h \circ \gamma \neq 0 \quad \text{for all } \gamma \in \text{SL}_2(\mathbb{Z}). \quad (2)$$

*Then  $h - c$  is not a modular unit for any  $c \in \mathbb{C}^\times$ .*

PROOF. Let us consider the holomorphic map between two compact Riemann surfaces

$$\begin{aligned} \varphi : X(N) &\rightarrow \mathbb{P}^1(\mathbb{C}) \\ \tau &\mapsto [h(\tau) : 1]. \end{aligned}$$

Since  $h$  is not a constant by (2),  $\varphi$  is surjective.

Now, let  $c \in \mathbb{C}^\times$ . Since  $\varphi$  is surjective and the values of  $\varphi$  at the cusps of  $X(N)$  are either  $[0 : 1]$  or  $[\infty : 1] = [1 : 0]$  by (2), there exists  $\tau_0 \in \mathbb{H}$  such that  $\varphi(\tau_0) = [c : 1]$ . This implies that  $h(\tau) - c$  has a zero at  $\tau = \tau_0$ , and hence  $h - c$  is not a modular unit.  $\square$

EXAMPLE 2.11. Let  $N \geq 2$  and  $\begin{bmatrix} r \\ s \end{bmatrix} \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ . Consider the Siegel function

$$h(\tau) = g_{\begin{bmatrix} r \\ s \end{bmatrix}}(\tau)^{12N},$$

which is a modular unit of level  $N$  by Lemma 2.5. Then we have the following properties:

- (i)  $h \circ \gamma = g_{t_\gamma \begin{bmatrix} r \\ s \end{bmatrix}}(\tau)^{12N}$  for any  $\gamma \in \text{SL}_2(\mathbb{Z})$  [6, Chapter 2 Proposition 1.3],
- (ii)  $\text{ord}_q h = 6N \cdot \mathbf{B}_2(\langle r \rangle)$ , where  $\mathbf{B}_2(x) = x^2 - x + 1/6$  is the second Bernoulli polynomial and  $\langle x \rangle$  is the fractional part of  $x$  such that  $0 \leq \langle x \rangle < 1$  for  $x \in \mathbb{R}$  [6, p.31],
- (iii)  $\mathbf{B}_2(x) \neq 0$  for all  $x \in \mathbb{Q}$ .

Thus  $h$  satisfies the assumption (2) in Proposition 2.10.

REMARK 2.12. If  $h$  does not satisfy the assumption (2), then  $h - c$  could be a modular unit for some constant  $c \in \mathbb{C}^\times$  (see Remark 3.4).

### 3 Integral closures in modular function fields

In this section, when  $N \equiv 0 \pmod{4}$  we shall investigate explicit generators of the integral closure  $\mathcal{O}_N$  of  $\mathbb{C}[j]$  in  $\mathbb{C}(X(N))$  by using the Weierstrass units.

For a lattice  $L = [\omega_1, \omega_2] = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$  in  $\mathbb{C}$  the Weierstrass  $\wp$ -function is defined by

$$\wp(z; L) = \frac{1}{z^2} + \sum_{\omega \in L - \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right) \quad (z \in \mathbb{C}).$$

LEMMA 3.1. *Let  $z, w \in \mathbb{C} - L$ . Then,  $\wp(z; L) = \wp(w; L)$  if and only if  $z \equiv \pm w \pmod{L}$ .*

PROOF. [10, Chapter IV §3].  $\square$

Let  $N \geq 2$ . For a vector  $\begin{bmatrix} r \\ s \end{bmatrix} \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$  we define

$$\wp_{\begin{bmatrix} r \\ s \end{bmatrix}}(\tau) = \wp(r\tau + s; [\tau, 1]) \quad (\tau \in \mathbb{H}),$$

which is a weakly holomorphic modular form of level  $N$  and weight 2 [7, Chapter 6]. More precisely, it satisfies the transformation formula

$$\wp_{\begin{bmatrix} r \\ s \end{bmatrix}}(\tau) \circ \gamma = (c\tau + d)^2 \wp_{t_\gamma \begin{bmatrix} r \\ s \end{bmatrix}}(\tau) \quad \text{for any } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}). \quad (3)$$

Hence the following function

$$(\wp_{\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}}(\tau) - \wp_{\begin{bmatrix} c_1 \\ d_1 \end{bmatrix}}(\tau)) / (\wp_{\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}}(\tau) - \wp_{\begin{bmatrix} c_2 \\ d_2 \end{bmatrix}}(\tau))$$

for  $\begin{bmatrix} a_k \\ b_k \end{bmatrix}, \begin{bmatrix} c_k \\ d_k \end{bmatrix} \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$  with  $\begin{bmatrix} a_k \\ b_k \end{bmatrix} \not\equiv \pm \begin{bmatrix} c_k \\ d_k \end{bmatrix} \pmod{\mathbb{Z}^2}$  ( $k = 1, 2$ ) is a modular unit of level  $N$  by Lemma 3.1, which is called a Weierstrass unit of level  $N$ .

We further define three functions on  $\mathbb{H}$

$$\begin{aligned} g_2(\tau) &= 60 \sum_{\omega \in [\tau, 1] - \{0\}} \omega^{-4}, \\ g_3(\tau) &= 140 \sum_{\omega \in [\tau, 1] - \{0\}} \omega^{-6}, \\ \Delta(\tau) &= g_2(\tau)^3 - 27g_3(\tau)^2, \end{aligned}$$

which are modular forms of level 1 and weight 4, 6 and 12, respectively [7, Chapter 3 Theorem 3].

For a positive integer  $N$ , let

$$\Gamma_1(N) = \{\gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \pmod{N}\},$$

and let  $X_1(N) = \bar{\Gamma}_1(N) \backslash \mathbb{H}^*$  be the corresponding modular curve where  $\bar{\Gamma}_1(N) = \Gamma_1(N) / \{\pm I_2\}$ .

LEMMA 3.2. (i) If  $N \geq 2$ , then  $\mathbb{C}(X_1(N)) = \mathbb{C}(j, (g_2g_3/\Delta)\wp_{\begin{bmatrix} 0 \\ 1/N \end{bmatrix}})$ .

(ii) If  $N \geq 2$ , then  $\mathbb{C}(X(N)) = \mathbb{C}(X_1(N))((g_2g_3/\Delta)\wp_{\begin{bmatrix} 1/N \\ 0 \end{bmatrix}})$ .

(iii)  $\mathbb{C}(X_1(4)) = \mathbb{C}(g_{\begin{bmatrix} 1/4 \\ 0 \end{bmatrix}}(4\tau)^{-8}g_{\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}(4\tau)^8)$ .

PROOF. (i), (ii) [2, Proposition 7.5.1].

(iii) [5, Table 2].

□

The modular curve  $X_1(4)$  is of genus 0 and has three inequivalent cusps, namely 0,  $1/2$  and  $i\infty$  [4, p.131]. Put

$$g_{1,4}(\tau) = g_{\begin{bmatrix} 1/4 \\ 0 \end{bmatrix}}(4\tau)^{-8} g_{\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}(4\tau)^8,$$

which is a primitive generator of  $\mathbb{C}(X_1(4))$  over  $\mathbb{C}$  by Lemma 3.2(iii). It then follows from [5, Theorem 6.5] that the map

$$\begin{aligned} X_1(4) = \overline{\Gamma}_1(4) \backslash \mathbb{H}^* &\rightarrow \mathbb{P}^1(\mathbb{C}) \\ \tau &\mapsto [g_{1,4}(\tau) : 1] \end{aligned}$$

is an isomorphism between compact Riemann surfaces. Moreover,  $g_{1,4}(\tau)$  has values 16, 0,  $\infty$  at the cusps  $\tau = 0, 1/2, i\infty$ , respectively ([4, Theorem 3(ii)] and [5, Table3]). Thus we claim that

$$g_{1,4} - c \text{ for } c \in \mathbb{C} \text{ is a modular unit (for } \Gamma_1(4)) \iff c = 16 \text{ or } 0. \quad (4)$$

**THEOREM 3.3.** *Let  $\mathcal{O}_{1,N}$  and  $\mathcal{O}_N$  be the integral closures of  $\mathbb{C}[j]$  in  $\mathbb{C}(X_1(N))$  and  $\mathbb{C}(X(N))$ , respectively. Assume that  $N \equiv 0 \pmod{4}$ .*

- (i)  $\mathcal{O}_{1,4} = \mathbb{C}[g_{1,4}, g_{1,4}^{-1}, (g_{1,4} - 16)^{-1}]$ .
- (ii)  $\mathcal{O}_{1,N} = \mathcal{O}_{1,4}[h_{1,N}]$ , where  $h_{1,N}(\tau) = (\wp_{\begin{bmatrix} 0 \\ 1/N \end{bmatrix}}(\tau) - \wp_{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}(\tau)) / (\wp_{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}(\tau) - \wp_{\begin{bmatrix} 0 \\ 1/4 \end{bmatrix}}(\tau))$ .
- (iii)  $\mathcal{O}_N = \mathcal{O}_{1,N}[h_N]$ , where  $h_N(\tau) = (\wp_{\begin{bmatrix} 1/N \\ 0 \end{bmatrix}}(\tau) - \wp_{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}(\tau)) / (\wp_{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}(\tau) - \wp_{\begin{bmatrix} 0 \\ 1/4 \end{bmatrix}}(\tau))$ .

**PROOF.** (i) Since  $g_{1,4}$  and  $g_{1,4} - 16$  are modular units in  $\mathbb{C}(X_1(4))$  by Lemma 3.2(iii) and (4), we get the inclusion  $\mathcal{O}_{1,4} \supseteq \mathbb{C}[g_{1,4}, g_{1,4}^{-1}, (g_{1,4} - 16)^{-1}]$ .

Conversely, let  $h \in \mathcal{O}_{1,4}$ . Then it is a rational function of  $g_{1,4}$  by Lemma 3.2(iii), namely  $h = P(g_{1,4})/Q(g_{1,4})$  for some polynomials  $P(X), Q(X) \in \mathbb{C}[X]$  which are relatively prime. If  $Q(X)$  has a linear factor other than  $g_{1,4}$  and  $g_{1,4} - 16$ , then  $h$  has a pole on  $\mathbb{H}$  by (4). Hence we obtain the reverse inclusion  $\mathcal{O}_{1,4} \subseteq \mathbb{C}[g_{1,4}, g_{1,4}^{-1}, (g_{1,4} - 16)^{-1}]$ . This proves (i).

(ii) Since  $h_{1,N} \in \mathcal{O}_{1,N}$  by Lemma 3.2(i) and the paragraph below Lemma 3.1, we have the inclusion  $\mathcal{O}_{1,N} \supseteq \mathcal{O}_{1,4}[h_{1,N}]$ .

Let  $f \in \mathcal{O}_{1,N}$ . Since

$$\begin{aligned} \mathbb{C}(X_1(N)) &= \mathbb{C}(j, (g_2 g_3 / \Delta) \wp_{\begin{bmatrix} 0 \\ 1/N \end{bmatrix}}) \text{ by Lemma 3.2(i)} \\ &= \mathbb{C}(X_1(4))((g_2 g_3 / \Delta) \wp_{\begin{bmatrix} 0 \\ 1/N \end{bmatrix}}) \text{ because } j \in \mathbb{C}(X_1(4)) \\ &= \mathbb{C}(X_1(4))((g_2 g_3 / \Delta)((\wp_{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}} - \wp_{\begin{bmatrix} 0 \\ 1/4 \end{bmatrix}})h_{1,N} + \wp_{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}})) \\ &= \mathbb{C}(X_1(4))(h_{1,N}) \\ &\quad \text{because } (g_2 g_3 / \Delta) \wp_{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}, (g_2 g_3 / \Delta) \wp_{\begin{bmatrix} 0 \\ 1/4 \end{bmatrix}} \in \mathbb{C}(X_1(4)) \text{ by Lemma 3.2(i),} \end{aligned}$$

$f$  can be written in the form

$$f = r_0 + r_1 h + r_2 h^2 + \cdots + r_{d-1} h^{d-1} \quad (5)$$

where  $h = h_{1,N}$ ,  $d = [\mathbb{C}(X_1(N)) : \mathbb{C}(X_1(4))]$  and  $r_0, r_2, \dots, r_{d-1} \in \mathbb{C}(X_1(4))$ . Multiplying both sides of the equation (5) by  $1, h, \dots, h^{d-1}$ , respectively, we attain a linear system (with unknowns  $r_0, r_1, \dots, r_{d-1}$ )

$$\begin{bmatrix} 1 & h & \cdots & h^{d-1} \\ h & h^2 & \cdots & h^d \\ \vdots & \vdots & \ddots & \vdots \\ h^{d-1} & h^d & \cdots & h^{2d-2} \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{d-1} \end{bmatrix} = \begin{bmatrix} f \\ hf \\ \vdots \\ h^{d-1}f \end{bmatrix}.$$

Taking the trace  $\text{Tr} (= \text{Tr}_{\mathbb{C}(X_1(N))/\mathbb{C}(X_1(4))})$  on both sides we achieve

$$\begin{bmatrix} \text{Tr}(1) & \text{Tr}(h) & \cdots & \text{Tr}(h^{d-1}) \\ \text{Tr}(h) & \text{Tr}(h^2) & \cdots & \text{Tr}(h^d) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Tr}(h^{d-1}) & \text{Tr}(h^d) & \cdots & \text{Tr}(h^{2d-2}) \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{d-1} \end{bmatrix} = \begin{bmatrix} \text{Tr}(f) \\ \text{Tr}(hf) \\ \vdots \\ \text{Tr}(h^{d-1}f) \end{bmatrix}. \quad (6)$$

Let  $T$  be the  $d \times d$  matrix in the left side of (6), and let  $c_1, c_2, \dots, c_d$  be the conjugates of  $h \in \mathbb{C}(X_1(N))$  over  $\mathbb{C}(X_1(4))$ . Then we get that

$$\begin{aligned} \det(T) &= \begin{vmatrix} \sum_{k=1}^d c_k^0 & \sum_{k=1}^d c_k^1 & \cdots & \sum_{k=1}^d c_k^{d-1} \\ \sum_{k=1}^d c_k^1 & \sum_{k=1}^d c_k^2 & \cdots & \sum_{k=1}^d c_k^d \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^d c_k^{d-1} & \sum_{k=1}^d c_k^d & \cdots & \sum_{k=1}^d c_k^{2d-2} \end{vmatrix} \\ &= \begin{vmatrix} c_1^0 & c_2^0 & \cdots & c_d^0 \\ c_1^1 & c_2^1 & \cdots & c_d^1 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{d-1} & c_2^{d-1} & \cdots & c_d^{d-1} \end{vmatrix} \begin{vmatrix} c_1^0 & c_1^1 & \cdots & c_1^{d-1} \\ c_2^0 & c_2^1 & \cdots & c_2^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_d^0 & c_d^1 & \cdots & c_d^{d-1} \end{vmatrix} \\ &= \prod_{1 \leq m < n \leq d} (c_m - c_n)^2 \quad \text{by the Van der Monde determinant formula.} \end{aligned}$$

On the other hand, any conjugate of  $h \in \mathbb{C}(X_1(N))$  over  $\mathbb{C}(X_1(4))$  is of the form

$$(\wp\left[\frac{a}{b/N}\right](\tau) - \wp\left[\frac{0}{1/2}\right](\tau)) / (\wp\left[\frac{0}{1/2}\right](\tau) - \wp\left[\frac{0}{1/4}\right](\tau)) \quad \text{for some } \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{Z}^2 - N\mathbb{Z}^2$$

owing to the fact  $\text{Gal}(\mathbb{C}(X_1(N))/\mathbb{C}(X_1(4))) \simeq \bar{\Gamma}_1(N)/\bar{\Gamma}_1(4)$ , the transformation formula (3) and Lemma 3.1. Moreover, we see that the function

$$(\wp\left[\frac{a}{b/N}\right](\tau) - \wp\left[\frac{c}{d/N}\right](\tau)) / (\wp\left[\frac{0}{1/2}\right](\tau) - \wp\left[\frac{0}{1/4}\right](\tau))$$

for  $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{Z}^2 - N\mathbb{Z}^2$  with  $\begin{bmatrix} a \\ b \end{bmatrix} \not\equiv \pm \begin{bmatrix} c \\ d \end{bmatrix} \pmod{N\mathbb{Z}^2}$  has no zeros and poles on  $\mathbb{H}$  by Lemma 3.1. This implies that  $\det(T)$  becomes a modular unit in  $\mathbb{C}(X_1(4))$ , in particular,  $\det(T)$  belongs to  $\mathcal{O}_{1,4}^\times$ . It then follows that  $r_0, r_1, \dots, r_{d-1} \in \mathcal{O}_{1,4}$ , and hence we deduce the inclusion  $\mathcal{O}_{1,N} \subseteq \mathcal{O}_{1,4}[h_{1,N}]$ . This completes the proof of (ii).

(iii) In like manner as in the proof of (ii) one can prove (iii).  $\square$



REMARK 3.4. Let

$$\theta_2(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i(n+1/2)^2 \tau}, \quad \theta_3(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} \quad \text{and} \quad \theta_4(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i n^2 \tau}$$

be the classical Jacobi theta functions, and let

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (7)$$

be the Dedekind-eta function. Then they satisfy the relations

$$\theta_2(\tau)^4 + \theta_4(\tau)^4 = \theta_3(\tau)^4, \quad (8)$$

and

$$\theta_2(2\tau) = 2\eta(4\tau)^2/\eta(2\tau) \quad \text{and} \quad \theta_4(2\tau) = \eta(\tau)^2/\eta(2\tau), \quad (9)$$

due to Jacobi [1, pp.27–29]. Furthermore, we have

$$g_{1,4}(\tau) = 16\theta_3(2\tau)^4/\theta_2(2\tau)^4$$

as a modular unit with  $\text{ord}_q(g_{1,4} \circ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}) = 0$  [5, Table3 and Theorem 6.2]. Hence we derive that

$$\begin{aligned} g_{1,4}(\tau) - 16 &= 16\theta_3(2\tau)^4/\theta_2(2\tau)^4 - 16 \\ &= 16\theta_4(2\tau)^4/\theta_2(2\tau)^4 \quad \text{by (8)} \\ &= \eta(\tau)^8/\eta(4\tau)^8 \quad \text{by (9)} \\ &= q^{-1} \prod_{n=1}^{\infty} (1 + q^n)^{-8} (1 + q^{2n})^{-8} \quad \text{by the definition (7).} \end{aligned}$$

Therefore,  $g_{1,4} - 16$  is indeed a modular unit.

COROLLARY 3.5. *Every weakly holomorphic modular function can be expressed as a sum of modular units (of higher level).*

PROOF. Let  $h$  be a weakly holomorphic modular function of level  $N$ . Since it belongs to  $\mathcal{O}_{4N/\text{gcd}(4,N)}$  by Proposition 2.3,  $h$  can be written as a sum of modular units of level  $4N/\text{gcd}(4, N)$  by Theorem 3.3. This completes the proof.  $\square$

Let  $k$  and  $N$  ( $\geq 1$ ) be integers. We denote the vector space of all weakly holomorphic modular forms of level  $N$  and weight  $k$  by  $\mathcal{M}_k^!(\Gamma(N))$ . Then we have a graded algebra

$$\mathcal{M}^!(\Gamma(N)) = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k^!(\Gamma(N))$$

with respect to weight  $k$ .

Now, define a Klein form

$$\mathfrak{k}_{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}(\tau) = (1/2\pi i) g_{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}(\tau) / \eta(\tau)^2$$

which belongs to  $\mathcal{M}_{-1}^!(\Gamma(8))$  [6, Chapter 3 Theorem 4.1]. It has no zeros and poles on  $\mathbb{H}$  by the expansion formulas (1) and (7).

THEOREM 3.6. For  $N \equiv 0 \pmod{8}$ , we get

$$\mathcal{M}^!(\Gamma(N)) = \mathcal{O}_N[\mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}, \mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}^{-1}] = \mathbb{C}[g_{1,4}, g_{1,4}^{-1}, (g_{1,4} - 16)^{-1}, h_{1,N}, h_N, \mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}, \mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}^{-1}]$$

where  $g_{1,4}$ ,  $h_{1,N}$  and  $h_N$  are functions described in Theorem 3.3.

PROOF. It is obvious that  $\mathcal{M}_0^!(\Gamma(N)) = \mathcal{O}_N$ .

If  $k \neq 0$ , then the following linear map

$$\begin{aligned} \varphi : \mathcal{O}_N &\rightarrow \mathcal{M}_k^!(\Gamma(N)) \\ h &\mapsto \mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}^{-k} h \end{aligned}$$

is an isomorphism, because  $\mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}^{-1} \in \mathcal{M}_1^!(\Gamma(8))$  and  $\mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}} \in \mathcal{M}_{-1}^!(\Gamma(8))$ . Thus  $\mathcal{M}_k^!(\Gamma(N)) = \mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}^{-k} \mathcal{O}_N$  as an  $\mathcal{O}_N$ -module. Therefore we attain from Theorem 3.3

$$\begin{aligned} \mathcal{M}^!(\Gamma(N)) &= \bigoplus_{k \in \mathbb{Z}} \mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}^{-k} \mathcal{O}_N \\ &= \mathcal{O}_N[\mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}, \mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}^{-1}] \\ &= \mathbb{C}[g_{1,4}, g_{1,4}^{-1}, (g_{1,4} - 16)^{-1}, h_{1,N}, h_N, \mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}, \mathfrak{k}_{\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}}^{-1}]. \end{aligned}$$

□

## 4 Siegel modular functions

In this section we shall show that if  $f(Z)$  is a Siegel modular function of degree  $g$  ( $\geq 2$ ) that has no zeros and poles on  $\mathbb{H}_g^{\text{diag}}$ , then  $f(Z)$  is a product of  $g$  modular units.

LEMMA 4.1. Let  $g$  and  $N$  be two positive integers  $\geq 2$ . If  $f(Z)$  is a Siegel modular function of degree  $g$  and level  $N$ , then the function

$$f(\text{diag}(\tau_1, \tau_2, \dots, \tau_g)) \quad (\text{diag}(\tau_1, \tau_2, \dots, \tau_g) \in \mathbb{H}_g^{\text{diag}}),$$

as a function of  $\tau_k$  ( $k = 1, 2, \dots, g$ ), is a meromorphic modular function of level  $N$ .

PROOF. Let

$$\gamma_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) \quad (k = 1, 2, \dots, g),$$

and set

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \text{diag}(a_1, a_2, \dots, a_g) & \text{diag}(b_1, b_2, \dots, b_g) \\ \text{diag}(c_1, c_2, \dots, c_g) & \text{diag}(d_1, d_2, \dots, d_g) \end{bmatrix},$$

where  $A, B, C, D$  are  $g \times g$  block matrices. Then we derive that

$$\begin{aligned}
{}^t\gamma J\gamma &= \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{because } A, B, C, D \text{ are diagonal} \\
&= \begin{bmatrix} CA - AC & CB - AD \\ DA - BC & DB - BD \end{bmatrix} \\
&= \begin{bmatrix} 0 & \text{diag}(c_1 b_1 - a_1 d_1, \dots, c_g b_g - a_g d_g) \\ \text{diag}(d_1 a_1 - b_1 c_1, \dots, d_g a_g - b_g c_g) & 0 \end{bmatrix} \\
&= J \quad \text{due to } \det(\gamma_k) = a_k d_k - b_k c_k = 1 \ (k = 1, 2, \dots, g),
\end{aligned}$$

from which we see that  $\gamma$  belongs to the group  $\text{Sp}_g(\mathbb{Z})$ .

And, for  $Z = \text{diag}(\tau_1, \tau_2, \dots, \tau_g) \in \mathbb{H}_g^{\text{diag}}$  we achieve that

$$\begin{aligned}
\gamma(Z) &= (AZ + B)(CZ + D)^{-1} \\
&= \text{diag}(a_1 \tau_1 + b_1, \dots, a_g \tau_g + b_g) \text{diag}(c_1 \tau_1 + d_1, \dots, c_g \tau_g + d_g)^{-1} \\
&= \text{diag}((a_1 \tau_1 + b_1)(c_1 \tau_1 + d_1)^{-1}, \dots, (a_g \tau_g + b_g)(c_g \tau_g + d_g)^{-1}) \\
&= \text{diag}(\gamma_1(\tau_1), \gamma_2(\tau_2), \dots, \gamma_g(\tau_g)). \tag{10}
\end{aligned}$$

On the other hand, assume that  $\gamma_k \equiv I_2 \pmod{N}$  for all  $k = 1, 2, \dots, g$ . Then  $\gamma \equiv I_{2g} \pmod{N}$ , and for  $Z = \text{diag}(\tau_1, \tau_2, \dots, \tau_g) \in \mathbb{H}_g^{\text{diag}}$  we have

$$\begin{aligned}
f(\text{diag}(\tau_1, \tau_2, \dots, \tau_g)) &= f(Z) \\
&= f(\gamma(Z)) \quad \text{since } f \text{ is of level } N \\
&= f(\text{diag}(\gamma_1(\tau_1), \gamma_2(\tau_2), \dots, \gamma_g(\tau_g))) \quad \text{by (10)}.
\end{aligned}$$

In particular, when  $k$  is fixed ( $k = 1, 2, \dots, g$ ) and  $\gamma_n = I_2$  for all  $n \neq k$ , we conclude that  $f(Z)$ , as a function of  $\tau_k$ , is a meromorphic modular function of level  $N$ .  $\square$

**THEOREM 4.2.** *Let  $g$  and  $N$  be two positive integers  $\geq 2$ , and let  $f(Z)$  be a Siegel modular function of degree  $g$  and level  $N$ . Then,  $f(Z)$  has no zeros and poles on  $\mathbb{H}_g^{\text{diag}}$  if and only if there exist modular units  $v_1(\tau), v_2(\tau), \dots, v_g(\tau) \in V_N$  such that*

$$f(\text{diag}(\tau_1, \tau_2, \dots, \tau_g)) = \prod_{k=1}^g v_k(\tau_k).$$

**PROOF.** The proof of “if” part is clear.

Conversely, assume that  $f(Z)$  has no zeros and poles on  $\mathbb{H}_g^{\text{diag}}$ . Let  $n (\geq 2)$  be the number of inequivalent cusps of  $X(N)$ . Since  $V_N/\mathbb{C}^\times$  is a free abelian group of rank  $n - 1$  by Proposition 2.7, there exist  $g_1(\tau), g_2(\tau), \dots, g_{n-1}(\tau) \in V_N$  such that  $V_N = \langle \mathbb{C}^\times, g_1, g_2, \dots, g_{n-1} \rangle$ . Thus  $f(\text{diag}(\tau_1, \tau_2, \dots, \tau_g))$ , as a function of  $\tau_g$ , can be written as

$$f(\text{diag}(\tau_1, \tau_2, \dots, \tau_g)) = c(\tau_1, \tau_2, \dots, \tau_{g-1}) \prod_{t=1}^{n-1} g_t(\tau_g)^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \tag{11}$$

where  $c : \mathbb{H}^{g-1} \rightarrow \mathbb{C}^\times$  and  $m_t : \mathbb{H}^{g-1} \rightarrow \mathbb{Z}$  are functions of  $\tau_1, \tau_2, \dots, \tau_{g-1}$  by Lemma 4.1 and the assumption.

Then we deduce that

$$\begin{aligned}
& \prod_{\gamma \in \bar{\Gamma}(1)/\bar{\Gamma}(N)} f(\text{diag}(\tau_1, \tau_2, \dots, \tau_{g-1}, \gamma(\tau_g))) \\
&= \prod_{\gamma \in \bar{\Gamma}(1)/\bar{\Gamma}(N)} \left( c(\tau_1, \tau_2, \dots, \tau_{g-1}) \prod_{t=1}^{n-1} g_t(\gamma(\tau_g))^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \right) \quad \text{by (11)} \\
&= c(\tau_1, \tau_2, \dots, \tau_{g-1})^d \prod_{t=1}^{n-1} \left( \prod_{\gamma \in \bar{\Gamma}(1)/\bar{\Gamma}(N)} g_t(\gamma(\tau_g)) \right)^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})}, \quad \text{where } d = |\bar{\Gamma}(1)/\bar{\Gamma}(N)| \\
&= c(\tau_1, \tau_2, \dots, \tau_{g-1})^d \prod_{t=1}^{n-1} N_{\mathbb{C}(X(N))/\mathbb{C}(X(1))}(g_t(\tau_g))^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \\
&\quad \text{due to the fact } \text{Gal}(\mathbb{C}(X(N))/\mathbb{C}(X(1))) \simeq \bar{\Gamma}(1)/\bar{\Gamma}(N) \\
&= c(\tau_1, \tau_2, \dots, \tau_{g-1})^d \prod_{t=1}^{n-1} c_t^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \quad \text{for some } c_1, c_2, \dots, c_{n-1} \in \mathbb{C}^\times \text{ by Remark 2.8,}
\end{aligned}$$

which is a modular unit of level  $N$  as a function of each  $\tau_k$  ( $k = 1, 2, \dots, g-1$ ) by Lemma 4.1. It follows from (11) that

$$\begin{aligned}
& f(\text{diag}(\tau_1, \tau_2, \dots, \tau_g))^d / \prod_{\gamma \in \bar{\Gamma}(1)/\bar{\Gamma}(N)} f(\text{diag}(\tau_1, \tau_2, \dots, \tau_{g-1}, \gamma(\tau_g))) \\
&= \left( c(\tau_1, \tau_2, \dots, \tau_{g-1}) \prod_{t=1}^{n-1} g_t(\tau_g)^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \right)^d / \left( c(\tau_1, \tau_2, \dots, \tau_{g-1})^d \prod_{t=1}^{n-1} c_t^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \right) \\
&= \prod_{t=1}^{n-1} (c_t^{-1} g_t(\tau_g)^d)^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})}.
\end{aligned}$$

Now, set this function to be  $h(\tau_1, \tau_2, \dots, \tau_g)$  which is a modular unit as a function of each  $\tau_k$  ( $k = 1, 2, \dots, g$ ).

On the other hand, when  $\tau_g \in \mathbb{H}$  is fixed, the image of the holomorphic function

$$\begin{aligned}
\varphi : \mathbb{H}^{g-1} &\rightarrow \mathbb{C}^\times \\
(\tau_1, \tau_2, \dots, \tau_{g-1}) &\mapsto h(\tau_1, \tau_2, \dots, \tau_g) = \prod_{t=1}^{n-1} (c_t^{-1} g_t(\tau_g)^d)^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})}
\end{aligned} \tag{12}$$

is a countable set, because  $m_t(\tau_1, \tau_2, \dots, \tau_{g-1})$  ( $t = 1, 2, \dots, n-1$ ) are integer-valued functions. Let  $\ell$  be an index in  $\{1, 2, \dots, g-1\}$  and suppose that  $\tau_1, \tau_2, \dots, \tau_{g-1}$  are fixed except for  $\tau_\ell$ . Then  $\varphi$  can be viewed as a holomorphic map from  $\mathbb{H}$  to  $\mathbb{C}^\times$  with respect to  $\tau_\ell$ . Since its image is a countable set as mentioned above, the modular unit  $h(\tau_1, \tau_2, \dots, \tau_g)$ , as a function of  $\tau_\ell$ , must be a constant by Lemma 2.9. This observation essentially indicates that the map  $\varphi$  defined on  $\mathbb{H}^{g-1}$  in (12) is in fact a constant, and hence the function  $h(\tau_1, \tau_2, \dots, \tau_g)$  of  $g$  variables is a function of  $\tau_g$ . Moreover, since  $g_1(\tau), g_2(\tau), \dots, g_{n-1}(\tau)$  form a basis for the free abelian group

$V_N/\mathbb{C}^\times$ , the integer-valued functions  $m_t(\tau_1, \tau_2, \dots, \tau_{g-1})$  ( $t = 1, 2, \dots, n-1$ ) should be fixed integers, say  $m_t$ . Thus if we set  $v_g(\tau) = \prod_{t=1}^{n-1} g_t(\tau)^{m_t} \in V_N$ , then we derive from (11) that

$$f(\text{diag}(\tau_1, \tau_2, \dots, \tau_g)) = c(\tau_1, \tau_2, \dots, \tau_{g-1})v_g(\tau_g). \quad (13)$$

The only property of  $f(\text{diag}(\tau_1, \tau_2, \dots, \tau_g))$  necessary to have (13) is that it is a meromorphic modular function of level  $N$  as a function of each  $\tau_k$  ( $k = 1, 2, \dots, g$ ). Now that  $c(\tau_1, \tau_2, \dots, \tau_{g-1})$  retains this property, if we apply the same argument to  $c(\tau_1, \tau_2, \dots, \tau_{g-1})$  instead of  $f(\text{diag}(\tau_1, \tau_2, \dots, \tau_g))$  and repeat this process over and over again, then we eventually reach the conclusion after  $(g-1)$  steps.  $\square$

EXAMPLE 4.3. Let  $g, N \geq 1$ . For  $\mathbf{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_g \end{bmatrix}, \mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_g \end{bmatrix} \in \mathbb{Q}^g$  we define a theta constant by

$$\Theta_{[\mathbf{r}]}(Z) = \sum_{\mathbf{n} \in \mathbb{Z}^g} e^{(t(\mathbf{n} + \mathbf{r})Z(\mathbf{n} + \mathbf{r})/2 + {}^t(\mathbf{n} + \mathbf{r})\mathbf{s})} \quad (Z \in \mathbb{H}_g),$$

where  $e(z) = e^{2\pi iz}$  for  $z \in \mathbb{C}$ . We further set

$$\Phi_{[\mathbf{r}]}(Z) = \Theta_{[\mathbf{r}]}(Z)/\Theta_{[\mathbf{0}]}(Z) \quad (Z \in \mathbb{H}_g),$$

which is a Siegel modular function of level  $2N^2$  [9, Proposition 7].

Now, we assume that  $g \geq 2$ ,  $Z' \in \mathbb{H}_{g-1}$  and  $\tau \in \mathbb{H}$ . Then we derive that

$$\begin{aligned} & \Theta_{[\mathbf{r}]} \left( \begin{bmatrix} Z' & 0 \\ 0 & \tau \end{bmatrix} \right) \\ &= \sum_{\mathbf{n} = \begin{bmatrix} n_1 \\ \vdots \\ n_g \end{bmatrix} \in \mathbb{Z}^g} e \left( \frac{1}{2} {}^t \begin{bmatrix} \mathbf{n}' + \mathbf{r}' \\ n_g + r_g \end{bmatrix} \begin{bmatrix} Z' & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} \mathbf{n}' + \mathbf{r}' \\ n_g + r_g \end{bmatrix} + {}^t \begin{bmatrix} \mathbf{n}' + \mathbf{r}' \\ n_g + r_g \end{bmatrix} \begin{bmatrix} \mathbf{s}' \\ s_g \end{bmatrix} \right), \quad \text{where } \mathbf{n}' = \begin{bmatrix} n_1 \\ \vdots \\ n_{g-1} \end{bmatrix} \\ &= \sum_{\mathbf{n}' \in \mathbb{Z}^{g-1}} \sum_{n_g \in \mathbb{Z}} e^{(t(\mathbf{n}' + \mathbf{r}')Z'(\mathbf{r}' + \mathbf{s}')/2 + (n_g + r_g)\tau(n_g + r_g)/2 + {}^t(\mathbf{n}' + \mathbf{r}')\mathbf{s}' + (n_g + r_g)s_g)} \\ &= \left( \sum_{\mathbf{n}' \in \mathbb{Z}^{g-1}} e^{(t(\mathbf{n}' + \mathbf{r}')Z'(\mathbf{r}' + \mathbf{s}')/2 + {}^t(\mathbf{n}' + \mathbf{r}')\mathbf{s}')} \right) \left( \sum_{n_g \in \mathbb{Z}} e^{((n_g + r_g)\tau(n_g + r_g)/2 + (n_g + r_g)s_g)} \right) \\ &= \Theta_{[\mathbf{r}']} (Z') \Theta_{\begin{bmatrix} r_g \\ s_g \end{bmatrix}} (\tau). \end{aligned} \quad (14)$$

Applying this argument inductively we obtain

$$\Phi_{[\mathbf{r}]}(\text{diag}(\tau_1, \tau_2, \dots, \tau_g)) = \prod_{k=1}^g \Phi_{\begin{bmatrix} r_k \\ s_k \end{bmatrix}}(\tau_k) \quad (\text{diag}(\tau_1, \tau_2, \dots, \tau_g) \in \mathbb{H}_g^{\text{diag}}).$$

On the other hand, it follows from the Jacobi triple product identity [3, (17.3)] and the definition (1) in §2 that

$$\Phi_{\begin{bmatrix} r \\ s \end{bmatrix}}(\tau) = \begin{cases} e((2rs + r - s)/4)g_{\begin{bmatrix} 1/2-r \\ 1/2-s \end{bmatrix}}(\tau)/g_{\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}(\tau) & \text{if } \begin{bmatrix} r \\ s \end{bmatrix} \in \mathbb{Q}^2 - (1/2 + \mathbb{Z})^2, \\ 0 & \text{if } \begin{bmatrix} r \\ s \end{bmatrix} \in (1/2 + \mathbb{Z})^2. \end{cases}$$

Therefore we conclude that  $\Phi_{[\mathbf{r}]}(\text{diag}(\tau_1, \tau_2, \dots, \tau_g))$  has no zeros and poles on  $\mathbb{H}_g^{\text{diag}}$ , or is identically zero.

## References

- [1] J. H. Bruinier, G. van der Geer, G. Harder and D. Zagier, *The 1-2-3 of Modular Forms*, Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Universitext, Springer-Verlag, Berlin, 2008.
- [2] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Grad. Texts in Math. 228, Springer-Verlag, New York, 2005.
- [3] N. J. Fine, *Basic Hypergeometric Series and Applications*, Mathematical Surveys and Monographs 27, Amer. Math. Soc., Providence, R. I., 1988.
- [4] C. H. Kim and J. K. Koo, *Arithmetic of the modular function  $j_{1,4}$* , Acta Arith. 84 (1998), no. 2, 129–143.
- [5] J. K. Koo and D. H. Shin, *On some arithmetic properties of Siegel functions*, Math. Zeit. 264 (2010), no. 1, 137–177.
- [6] D. Kubert and S. Lang, *Modular Units*, Grundlehren der mathematischen Wissenschaften 244, Springer-Verlag, New York-Berlin, 1981.
- [7] S. Lang, *Elliptic Functions*, 2nd edn, Grad. Texts in Math. 112, Springer-Verlag, New York, 1987.
- [8] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten and Princeton University Press, Princeton, N. J., 1971.
- [9] G. Shimura, *Theta functions with complex multiplication*, Duke Math. J. 43 (1976), no. 4, 673–696.
- [10] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Grad. Texts in Math. 106, Springer-Verlag, New York, 1992.

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